Near-horizon symmetries of extremal black holes

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Abstract

Recent work has demonstrated an attractor mechanism for extremal rotating black holes subject to the assumption of a near-horizon $SO(2,1)$ symmetry. We prove the existence of this symmetry for any extremal black hole with the same number of rotational symmetries as known four and five dimensional solutions (including black rings). The result is valid for a general two-derivative theory of gravity coupled to abelian vectors and uncharged scalars, allowing for a non-trivial scalar potential. We prove that it remains valid in the presence of higher-derivative corrections. We show that $SO(2,1)$-symmetric near-horizon solutions can be analytically continued to give $SU(2)$-symmetric black hole solutions. For example, the near-horizon limit of an extremal 5D Myers-Perry black hole is related by analytic continuation to a non-extremal cohomogeneity-1 Myers-Perry solution.

1 Introduction

The "attractor mechanism" is the phenomenon that the entropy of an extremal black hole cannot depend on any moduli of the theory. It was initially discovered for supersymmetric black holes [1, 2, 3], then realized that it still applies in the presence of certain higher-derivative corrections [4, 5, 6], and most recently extended to non-supersymmetric black holes [7, 8, 9]. This has led to an explanation [10, 11] of the success of string theory calculations of the entropy of certain non-supersymmetric extremal black holes [12, 13, 14, 15, 16, 17].

Most studies of the attractor mechanism have concerned static, spherically symmetric, black holes. However, there has been recent interest in extending this to more general extremal black holes [18].

Any extremal black hole admits a near-horizon limit [19]. For known solutions, the isometry group of the black hole is enhanced in this limit. For example, the near-horizon geometry of the extremal Kerr black hole is [20]

$$ds^2 = \frac{(1+\cos^2\theta)}{2}\left[-\frac{r^2}{r_0^2}dv^2 + 2dvdr + r_0^2 d\theta^2\right] + \frac{2r_0^2\sin^2\theta}{1+\cos^2\theta}\left(d\phi + \frac{r}{r_0^2}dv\right)^2 \tag{1}$$

where $r_0 > 0$. The first two terms in square brackets are the metric of 2d anti-de Sitter space (AdS), which has isometry group $O(2,1)$. In fact (as we shall explain later), $O(2,1)$ extends to a symmetry of the full metric, so the full isometry group is $O(2,1) \times U(1)$ where the $U(1)$ arises from the axisymmetry of the black hole (generated by $\partial/\partial\phi$) [20]. Other 4d examples are considered in [18], with the conclusion that they also have $O(2,1) \times U(1)$ isometry group in the near-horizon limit.

Similarly, the near-horizon geometry of the extremal 5d Myers-Perry [21] black hole has $O(2,1) \times U(1)^2$ isometry group [20], where the $U(1)^2$ arises from the two rotational symmetries of this black hole. The original 2-parameter black ring solution [22] does not admit an extremal limit but its 3-parameter generalization [23] does, as does the dipole ring solution [24]. We shall see that the near-horizon geometries of these extremal solutions also have $O(2,1) \times U(1)^2$ symmetry.

In all of these examples, the $O(2,1)$ symmetry arises because the near-horizon geometry involves a fibration over AdS_2. For solutions with non-trivial Maxwell fields, the Maxwell field strengths are invariant only under the $SO(2,1)$ subgroup of $O(2,1)$ that preserves orientation in AdS_2. Hence the full near-horizon solution has symmetry group $SO(2,1) \times U(1)^{D-3}$ for $D = 4,5$.

If one *assumes* the existence of this $SO(2,1)$ symmetry in general then one can extend the attractor mechanism beyond the static, spherically symmetric case to general extremal black holes [18]. At first sight, the assumption of $SO(2,1)$ symmetry appears rather strong since, as we shall explain, a general near-horizon geometry possesses only a 2d non-abelian isometry group. However, we shall show that $SO(2,1)$ emerges dynamically, as a consequence of the Einstein equation, subject to the assumption that the black hole in question admits the same number $(D-3)$ of rotational symmetries as known black hole solutions in $D = 4,5$ dimensions.

We shall work with a general 2-derivative theory describing Einstein gravity coupled to abelian vectors A^I $(I = 1 \dots N)$ and uncharged scalars ϕ^A $(A = 1 \dots M)$ in $D = 4,5$ dimensions, with action

$$S = \int d^D x \sqrt{-g} \left(R - \frac{1}{2} f_{AB}(\phi) \partial_\mu \phi^A \partial^\mu \phi^B - V(\phi) - \frac{1}{4} g_{IJ}(\phi) F^I_{\mu\nu} F^{J\mu\nu} \right) + S_{\text{top}}, \tag{2}$$

where $F^I \equiv dA^I$, $V(\phi)$ is an arbitrary scalar potential (which allows for a cosmological constant), and

$$S_{\text{top}} = \frac{1}{2} \int h_{IJ}(\phi) F^I \wedge F^J \qquad \text{if } D = 4, \tag{3}$$

or

$$S_{\text{top}} = \frac{1}{6} \int C_{IJK} F^I \wedge F^J \wedge A^K \qquad \text{if } D = 5, \tag{4}$$

where C_{IJK} are constants.

This encompasses many theories of interest, e.g., vacuum gravity with a cosmological constant, Einstein-Maxwell theory, and various (possibly gauged) supergravity theories arising from compactification from ten or eleven dimensions. Furthermore, we shall not restrict attention only to asymptotically flat black holes. For example, our results will apply equally to asymptotically anti-de Sitter black holes. The first main result of this paper can be summarised in the following:

Theorem 1. Consider an asymptotically flat, or anti-de Sitter, extremal black hole solution of the above theory. Assume that it has $D-3$ rotational symmetries. Then the near-horizon limit of this solution has a global $G_3 \times U(1)^{D-3}$ symmetry, where G_3 is either $SO(2,1)$ or (the orientation-preserving subgroup of) the 2d Poincaré group. The Poincaré-symmetric case is excluded if $f_{AB}(\phi)$ and $g_{IJ}(\phi)$ are positive definite, the scalar potential is non-positive, and the horizon topology is not T^{D-2}.

Remarks:

1. The asymptotic boundary conditions are only required at one point in the proof, where we use the fact that the generator of each rotational symmetry must vanish somewhere in the

asymptotic region (on the "axis" of the symmetry) to constrain the Maxwell fields. The theorem is true for any asymptotic boundary conditions with this property.[1]

2. The existence of a single rotational symmetry seems reasonable because of the "stationary implies axisymmetric" theorem,[2] although this has only been proved for *non-extremal* rotating black holes. There is no general argument for the existence of *two* rotational symmetries in $D = 5$ but all known solutions have this property.

3. We will show that generic orbits of the symmetry group have the structure of T^{D-3} fibred over over a 2d maximally symmetric space, i.e., AdS_2, dS_2 or $R^{1,1}$. AdS_2 and dS_2 give $SO(2,1)$ symmetry whereas $R^{1,1}$ gives Poincaré symmetry. We can exclude the dS_2 and $R^{1,1}$ cases subject to the additional assumptions mentioned, which ensure that the theory obeys the strong energy condition. This encompasses many theories of interest e.g. theories for which the scalars are all moduli, or various gauged supergravity theories. The assumption that the horizon topology is non-toroidal (which is not needed if the scalar potential is strictly negative) seems reasonable because of the black hole topology theorem, which has been proved for Einstein gravity with a variety of asymptotic boundary conditions and restrictions on the energy-momentum tensor.[3]

Much of the interest in the attractor mechanism stems from the fact that it applies to any local, generally-covariant, theory, not just second-order gravity [7]. Therefore it is important to examine how higher-derivative corrections affect our result. Our second main result is the following:

Theorem 2. Consider a general theory of gravity coupled to abelian vectors A^I and uncharged scalars ϕ^A with action

$$S = S_2 + \sum_{m \geq 1} \lambda^m \int \sqrt{-g}\mathcal{L}_m, \tag{5}$$

where S_2 is the 2-derivative action above, λ is a coupling constant, and $\mathcal{L}_m$ is constructed by contracting (derivatives of) the Riemann tensor, volume form, scalar fields and Maxwell fields in such a way that the action is diffeomorphism and gauge-invariant. Consider an extremal black hole solution of this theory obeying the same assumptions as in Theorem 1. Assume that there is a regular horizon when $\lambda = 0$ with $SO(2,1) \times U(1)^{D-3}$ near-horizon symmetry (as guaranteed by Theorem 1), and that the near-horizon solution is analytic in λ. Then the near-horizon solution has $SO(2,1) \times U(1)^{D-3}$ symmetry to all orders in λ.

Hence our result is stable with respect to higher-derivative corrections. However, it does not apply to so-called "small" black holes [30, 31, 32, 33, 34], for which existence of a horizon depends on the higher-derivative terms in the action, i.e., it requires $\lambda \neq 0$.

The above theorems are proved in section 2. Section 3 discusses some examples of 5-dimensional near-horizon geometries. In particular, we discuss the near-horizon geometries of extremal Myers-Perry black holes and black rings. The near-horizon geometry of an extremal vacuum black ring turns out the be the same as that of an extremal boosted Kerr black string. We shall see that the $SO(2,1) \times U(1)^2$-invariant near-horizon geometries of Myers-Perry and black ring solutions can be analytically

[1] A $D = 5$ theory with Kaluza-Klein boundary conditions could violate this condition (if one of the rotational Killing fields were tangent to the Kaluza-Klein circle at infinity then it would not vanish anywhere in the asymptotic region). However, in this case one could simply apply our theorem to the $D = 4$ theory resulting from dimensional reduction.

[2] That has been proved for $D = 4$ Einstein gravity coupled to "reasonable" matter (obeying the weak energy condition with hyperbolic equations of motion) and asymptotically flat boundary conditions [25]. It has recently been extended to $D > 4$ and asymptotically anti-de Sitter boundary conditions [26].

[3] In $D = 4$ it has been proved for matter obeying the null energy condition and asymptotically flat [27] or asymptotically anti-de Sitter [28] boundary conditions. In $D = 5$ it has been proved for matter obeying the dominant energy condition and asymptotically flat boundary conditions [29].

continued to give stationary solutions with $SU(2) \times U(1) \times R$ symmetry, where R denotes time translations. For example, the near-horizon geometry of a cohomogeneity-2 Myers-Perry solution can be analytically continued to give a non-extremal cohomogeneity-1 (equal angular momenta) Myers-Perry solution. Similarly, the near-horizon geometry of an extremal dipole ring can be continued to give a Kaluza-Klein black hole. Finally, we determine the most general $SO(2,1)\times U(1)^{D-3}$-symmetric vacuum near-horizon geometry by exploiting the fact that the analytically continued version of this problem is to find the general stationary, spherically symmetric solution of Kaluza-Klein theory, which was solved in [35].

2 $SO(2,1)$ symmetry

2.1 Near-horizon limit

The event horizon of a stationary, non-extremal, black hole must be a Killing horizon. We shall assume that this is also true for extremal black holes. In the neighbourhood of the horizon, one can introduce Gaussian null coordinates (v, r, x^a), in which the metric takes the form (see e.g. [36])

$$ds^2 = r^2 F(r,x) dv^2 + 2dvdr + 2rh_a(r,x)dvdx^a + \gamma_{ab}(r,x)dx^a dx^b, \tag{6}$$

where $\partial/\partial v$ is Killing, the horizon is at $r = 0$ and x^a are coordinates on a $(D-2)$-dimensional spatial cross-section of the horizon. The functions F, h_a and γ_{ab} are continuous functions of r. Extremality implies that g_{vv} is $\mathcal{O}(r^2)$.

The *near-horizon limit* is defined by [19]

$$v \to v/\epsilon, \qquad r \to \epsilon r \tag{7}$$

and $\epsilon \to 0$, after which we obtain the near-horizon geometry

$$ds^2 = r^2 F(x) dv^2 + 2dvdr + 2rh_a(x)dvdx^a + \gamma_{ab}(x)dx^a dx^b, \tag{8}$$

with $F(x) \equiv F(0,x)$ etc. The near-horizon geometry is invariant under $v \to v +$ const (generated by $\partial/\partial v$) and also under the transformation defined by equation (7) (generated by $v\partial/\partial v - r\partial/\partial r$). These symmetries generate a two-dimensional non-abelian isometry group G_2. The orbits of this group are two-dimensional if $r \neq 0$ and one-dimensional if $r = 0$.

The main point of this paper is to demonstrate that the non-abelian G_2 symmetry group is enhanced to a larger $O(2,1)$ symmetry dynamically as a consequence of the Einstein equations.

2.2 Static black holes

In the static case, the $O(2,1)$ symmetry can be understood kinematically. For a static black hole, the generator of time translations must be null on the event horizon so the near-horizon geometry is a Killing horizon of this Killing vector field and hence $V \equiv \partial/\partial v$ must be hypersurface-orthogonal: $V \wedge dV = 0$. Hence the near-horizon geometry is also static. We then have

Lemma 0. A static near-horizon geometry is locally a warped product of a 2d maximally symmetric space-time with a compact $(D-2)$-manifold.

Proof. $V \wedge dV = 0$ if, and only if, $dh = 0$ and $dF = Fh$. Therefore, locally there exists a function $\lambda(x)$ such that $h = d\lambda$ and $F = F_0 \exp(\lambda)$. Now define $R = r\exp(\lambda)$. In coordinates (v, R, x) the near-horizon geometry is

$$ds^2 = e^{-\lambda(x)} \left(F_0 R^2 dv^2 + 2dvdR \right) + \gamma_{ab}(x)dx^a dx^b. \tag{9}$$

The terms in brackets describe a 2d maximally symmetric spacetime: de Sitter if $F_0 > 0$, Minkowski if $F_0 = 0$, and anti-de Sitter if $F_0 < 0$. γ_{ab} is the metric on a spatial cross-section of the horizon, which, for a black hole, is necessarily compact.

The word "locally" can be deleted if the horizon is simply connected. For a static black hole V must be timelike outside the horizon. After taking the near-horizon limit, this gives $F_0 \leq 0$ so the de Sitter case is excluded. So in general, the near-horizon geometry of a static extremal black hole is locally a warped product of AdS_2 or $R^{1,1}$ with a compact $(D-2)$-manifold. Hence there is a local $O(2,1)$ symmetry if $F_0 < 0$ and a local 2d Poincaré symmetry if $F_0 = 0$. The orbits of these isometry groups are 2-dimensional. The symmetry is global if the horizon is simply connected.

It is possible for a *non-static* black hole to have a static near-horizon geometry. Indeed, this is what happens for supersymmetric black rings, which have near-horizon geometry locally isometric to $AdS_3 \times S^2$ [37]. For these solutions, one finds that $F_0 = 0$ and hence there is a local Poincaré symmetry (it is only local because the horizon is not simply connected). The symmetry acts on the flat slices of AdS_3 written in "horospherical" coordinates. In this case, the Poincaré symmetry is obviously a subgroup of a much larger local symmetry group, and, as we shall see later, there is also a (global) $O(2,1)$ symmetry present. In the next section (and in the Appendix), we shall argue that, for black holes with the same amount of rotational symmetry as known solutions, the existence of a near-horizon Poincaré symmetry can *only* arise in this way, i.e., there will always be a global $O(2,1)$ symmetry in addition to the local Poincaré symmetry.

2.3 Rotational symmetries

If a stationary, non-extremal, black hole is *rotating*, i.e., if the stationary Killing field is not null on the event horizon, then it must be axisymmetric, i.e., it must admit a rotational $U(1)$ symmetry [25, 26]. Assuming that this is also true in the extremal case, the near-horizon metric (8) must also admit a $U(1)$ symmetry. Hence, on the basis of what has been proved for general black holes, we can expect an extremal rotating black hole to possess a near-horizon $G_2 \times U(1)$ symmetry.

For $D = 4$, the $G_2 \times U(1)$ symmetry implies that the near-horizon geometry is cohomogeneity-1. It turns out that the same is true for all *known* extremal black holes in $D = 5$. The reason is that all such black holes admit *two* rotational symmetries. It is not known whether this should be true in general, or whether it is an "accidental" property of the known solutions. In any case, in $D = 5$ we are going to restrict attention to black holes for which this is true. Hence we assume that there is a $U(1)^2$ rotational symmetry, and therefore the near-horizon geometry has $G_2 \times U(1)^2$ isometry group whose generic orbits are 4-dimensional, so the near-horizon geometry is cohomogeneity-1.

For the sake of generality, we shall consider a D-dimensional near-horizon geometry with a $G_2 \times U(1)^{D-3}$ isometry whose generic orbits are $(D-1)$-dimensional. For $D > 5$, some of the $U(1)$ factors must be translational, rather than rotational, symmetries as the rotation group $SO(D-1)$ admits a $U(1)^{D-3}$ subgroup only for $D = 4, 5$. The most natural interpretation for $D > 5$ is that we are considering a black brane rather than a black hole, with some worldvolume directions wrapped on a torus to give a black hole after reduction to $D = 4$ or $D = 5$.

If $D = 4$ then the only (compact) horizon topologies consistent with the existence of a global rotational Killing field are S^2 and T^2. If $D = 5$ then then only possibilities consistent with two rotational Killing fields are S^3 (or a quotient), $S^1 \times S^2$ and T^3 [38].

The existence of the $U(1)^{D-3}$ symmetry allows us to introduce coordinates $x^a = (\rho, x^i)$, $i = 1 \dots D-3$, such that $\partial/\partial x^i$ are Killing, and

$$\gamma_{ab} dx^a dx^b = d\rho^2 + \gamma_{ij}(\rho) dx^i dx^j. \tag{10}$$

For toroidal topology, γ_{ij} is non-degenerate and periodic in ρ. For non-toroidal topology, the range of ρ is a finite interval and γ_{ij} degenerates at the endpoints of this interval, where one of the Killing

fields vanishes. For S^2 or $S^1 \times S^2$ topology, it is the same Killing field that vanishes at each endpoint but for S^3 topology, it is a different Killing field at each endpoint [38].

Define a positive function $\Gamma(\rho)$ by

$$h_\rho = -\frac{\Gamma'}{\Gamma}, \tag{11}$$

and functions $k_i(\rho)$ by

$$h_i = \Gamma^{-1} k_i. \tag{12}$$

We can now perform a coordinate change $r \to \Gamma(\rho) r$ to bring the near-horizon metric to the form

$$ds^2 = r^2 A(\rho) dv^2 + 2\Gamma(\rho) dv dr + d\rho^2 + \gamma_{ij}(\rho) \left(dx^i + k^i(\rho) r dv\right) \left(dx^j + k^j(\rho) r dv\right), \tag{13}$$

where $k^i \equiv \gamma^{ij} k_j$. We are now ready for

Lemma 1. Consider a near-horizon geometry with symmetry $G_2 \times U(1)^{D-3}$. Introduce coordinates (v, r, ρ, x^i) as above. If the ρi and ρv components of the Ricci tensor vanish then k^i is constant and $A(\rho) = A_0 \Gamma(\rho)$ for some constant A_0. The near-horizon metric is

$$ds^2 = \Gamma(\rho) \left[A_0 r^2 dv^2 + 2 dv dr\right] + d\rho^2 + \gamma_{ij}(\rho) \left(dx^i + k^i r dv\right) \left(dx^j + k^j r dv\right). \tag{14}$$

Proof. Explicit calculation gives (a prime denotes a derivative with respect to ρ)

$$R_{\rho i} = \frac{1}{2\Gamma} \gamma_{ij} \left(k^j\right)', \tag{15}$$

$$R_{\rho v} = \frac{r}{\Gamma} \left[A' - \frac{\Gamma'}{\Gamma} A + \left(k^i\right)' k_i\right]. \tag{16}$$

Hence $R_{\rho i} = 0$ implies $(k^i)' = 0$ and then $R_{\rho v} = 0$ implies $A = A_0 \Gamma$.

The part of the metric (14) in square brackets is the metric of a 2d maximally symmetric space-time M_2: de Sitter if $A_0 > 0$, Minkowski if $A_0 = 0$ and anti-de Sitter if $A_0 < 0$. The next lemma shows that all symmetries of this 2d space-time extend to symmetries of the full near-horizon metric (14).

Lemma 2. The metric (14) has an isometry group $\hat{G}_3 \times U(1)^{D-3}$ where the 3-dimensional group $\hat{G}_3$ is the 2d Poincaré group if $A_0 = 0$ or $O(2,1)$ if $A_0 \neq 0$. The orbits of $\hat{G}_3$ are 3-dimensional if $k^i \neq 0$ and 2-dimensional if $k^i = 0$.

Proof. M_2 has a 3-dimensional isometry group $\hat{G}_3$. We need to show that these isometries extend to the rest of the metric. This is trivial if $k^i = 0$ (in which case the orbits of $\hat{G}_3$ are obviously 2-dimensional), so assume $k^i \neq 0$. The volume form of M_2 is $dr \wedge dv = d(r dv)$. This volume form is invariant up to a sign under $\hat{G}_3$. Hence under an isometry in $\hat{G}_3$ we must have $r dv \to \pm(r dv + d\phi)$ for some function $\phi(v, r)$. Since k^i is constant, we can compensate for this shift by a $U(1)$ transformation $x^i \to \pm(x^i - k^i \phi(v, r))$, so the full metric is invariant under an $\hat{G}_3 \times U(1)^{D-3}$ symmetry. The orbits of $\hat{G}_3$ are 3-dimensional because of this shift in x^i.

We shall prove below (and in the Appendix) that the $A_0 \geq 0$ case can be ruled out subject to the additional assumptions on the theory listed in Theorem 1. Therefore we are mainly interested in $A_0 < 0$.

It will be useful to have explicit expressions for the discrete symmetries of AdS_2. To do this we transform the AdS_2 to global coordinates, in such a way to make the enhancement of symmetry

manifest in the full near-horizon metric. This can be achieved by the transformation $(v, r, x^i) \to (T, Y, \chi^i)$ defined by:

$$\begin{pmatrix} r \\ g^2 v r \end{pmatrix} = \begin{pmatrix} -Y + g^{-1}(1+g^2Y^2)^{1/2}\sin(gT) \\ (1+g^2Y^2)^{1/2}\cos(gT) - 1 \end{pmatrix} \tag{17}$$

and

$$dx^i + k^i r dv = d\chi^i + k^i Y dT, \tag{18}$$

where for clarity we have written $A_0 = -g^2$ since we are concerned with $A_0 \leq 0$ (these equations are also valid for Poincaré symmetry ($A_0 = 0$) if one takes the limit $g \to 0$ with T, Y held fixed). Note that equation (18) is integrable since $dv \wedge dr = dT \wedge dY$, and also that $\partial/\partial x^i = \partial/\partial \chi^i$. In these coordinates the near-horizon geometry is

$$ds^2 = \Gamma(\rho)\left[-\left(1+g^2Y^2\right)dT^2 + \left(1+g^2Y^2\right)^{-1}dY^2\right] + d\rho^2 + \gamma_{ij}(\rho)\left(d\chi^i + k^iYdT\right)\left(d\chi^j + k^jYdT\right). \tag{19}$$

It is clear that this near-horizon metric exhibits the discrete isometries

$$P_1 : (T, \chi^i) \to (-T, -\chi^i), \qquad P_2 : (T, Y) \to (-T - Y) \tag{20}$$

which are inherited from the discrete T and PT isometries of AdS_2 respectively. P_1 is in $O(2,1)$ but not $SO(2,1)$. P_2 is in $SO(2,1)$ but not continuously connected to the identity.

We end this section by examining when the near-horizon geometry (14) is *static*. This occurs if, and only if (i) $k = 0$ or (ii) $k^2 = -A_0\Gamma$ with $A_0 < 0$, and $k_i = \Gamma \bar{k}_i$ where $\bar{k}_i$ is constant. In case (i) it is obvious that Lemma 2 is a special case of Lemma 0. Case (ii) is more interesting. In this case, we can choose our coordinates x^i so that $k = g\partial/\partial x^1$ and (by shifting x^1 if necessary) $\bar{k} \propto dx^1$. Split the coordinates as $x^i = (x^1, x^A)$, $A = 2 \ldots D-3$ (for $D \geq 5$). Then we have $\gamma_{11} = k^2/g^2 = \Gamma$ and $\gamma_{1A} = 0$. Hence $k = g\Gamma dx^1$. The metric is

$$ds^2 = \Gamma(\rho)\left[-g^2r^2dv^2 + 2dvdr + (dx^1 + grdv)^2\right] + d\rho^2 + \gamma_{AB}(\rho)dx^A dx^B, \tag{21}$$

The metric in square brackets is locally isometric to AdS_3. Hence in this case, the near-horizon geometry has local isometry group $O(2,2) \times U(1)^{D-4}$, where $O(2,2)$ has 3d orbits. $O(2,2)$ is only local because x^1 must be identified for the horizon to be compact. Globally, this breaks $O(2,2) \sim O(2,1) \times O(2,1)$ to the $O(2,1) \times U(1)$ guaranteed by Lemma 2.

2.4 General second order theory

As mentioned above, if $D > 5$ then some of the Killing directions must parameterize Kaluza-Klein directions so given a theory in $D > 5$ dimensions we can work in a dimensionally reduced theory with $D = 4$ or $D = 5$. We assume that this is of the type described in the Introduction. We can now present the proof of Theorem 1 stated in the Introduction. The method is first to show that a near-horizon solution of this theory will satisfy the assumptions of Lemma 1, and hence (Lemma 2) the metric will possess enhanced isometry group $\hat{G}_3$. Then we show that the other fields also exhibit symmetry enhancement, although only with respect to the subgroup G_3 of $\hat{G}_3$ that preserves orientation in M_2.

To satisfy the assumptions of Lemma 1, we need to show that $T_{\rho i} = T_{\rho v} = 0$ in the near-horizon limit for any extremal black hole solution of this theory, where $T_{\mu\nu}$ is the energy-momentum tensor. We assume that the matter fields are invariant with respect to the Killing fields $\partial/\partial v$ and $\partial/\partial x^i$. Hence, after taking the near-horizon limit, the scalar fields are functions of ρ only. The scalar kinetic and potential terms make a vanishing contribution to $T_{\rho i}$ since $g_{AB}(\phi)\partial_\rho\phi^A\partial_i\phi^B$ and $g_{\rho i}$ are zero. Similarly there is no scalar field contribution to $T_{\rho v}$.

Turning to the vector fields, we first make use of a standard result: if X and Y are commuting Killing vector fields that preserve a closed 2-form F then $F_{\mu\nu}X^\mu Y^\nu$ is constant. Take X to be a rotational symmetry. For conventional asymptotic boundary conditions (e.g. asymptotically flat or asymptotically anti-de Sitter), X must vanish somewhere in the full black hole space-time (on the "axis" of rotational symmetry). Hence $F_{\mu\nu}X^\mu Y^\nu \equiv 0$. Taking $X = \partial/\partial x^i$, and $Y = \partial/\partial x^j$ or $\partial/\partial v$, we conclude that F^I_{ij} and F^I_{vi} must vanish. Using this, the near-horizon limit of the Maxwell field must have the form[4]

$$F^I = F^I_{vr}(\rho) dv \wedge dr + r\tilde{F}^I_{v\rho}(\rho) dv \wedge d\rho + F^I_{\rho i}(\rho) d\rho \wedge dx^i. \tag{22}$$

Imposing the Bianchi identity $dF^I = 0$ implies $(F^I_{vr})' = \tilde{F}^I_{v\rho}$. One then finds (using the metric (13)) that

$$\left[d\left(g_{IJ}(\phi) \star F^J\right)\right]_{rvi_1\ldots i_{D-3}} = \sqrt{\gamma} g_{IJ}(\phi)\left(\tilde{F}^I_{v\rho} + k^i(\rho) F^I_{\rho i}\right). \tag{23}$$

The equation of motion for A^I says that this must be proportional to $d(h_{IJ}(\phi)F^J)_{rvi_1}$ if $D = 4$ or to $C_{IJK}(F^J \wedge F^K)_{rvi_1 i_2}$ if $D = 5$ but it is easy to see that both of these terms vanish. Hence the Maxwell equation implies that

$$\tilde{F}^I_{v\rho} = -k^i(\rho) F^I_{\rho i}. \tag{24}$$

Substituting this back into (22) gives

$$F^I = F^I_{vr}(\rho) dv \wedge dr + F^I_{\rho i}(\rho) d\rho \wedge \left(dx^i + k^i(\rho) r dv\right). \tag{25}$$

It is then easy to see that the Maxwell fields makes a vanishing contribution to $T_{\rho i}$ and $T_{\rho v}$. Hence we have satisfied the conditions of Lemma 1 so we must have $A = A_0\Gamma$ and k^i is constant. Therefore, from Lemma 2, the metric exhibits an enhanced isometry group $\hat{G}_3$.

Converting (25) to global coordinates gives

$$F^I = F^I_{TY}(\rho) dT \wedge dY + F^I_{\rho i}(\rho) d\rho \wedge \left(d\chi^i + k^i Y dT\right). \tag{26}$$

It is now obvious that the Maxwell fields inherit all the *continuous* enhanced symmetries of the metric, as well as the discrete symmetry P_2 of (20). However, under the discrete symmetry P_1, we have

$$F^I \to -F^I. \tag{27}$$

Hence, although the metric and scalars are invariant under $\hat{G}_3$, the Maxwell fields are only invariant under the subgroup of $\hat{G}_3$ that preserves orientation in M_2, which we shall denote as G_3. If $A_0 < 0$ then $G_3 = SO(2,1)$.

We can now rule out the $A_0 \geq 0$ case, which we analyze in the Appendix (note this includes the Poincaré symmetric case $A_0 = 0$). If the matrices f_{AB} and g_{IJ} occurring in the scalar and vector kinetic terms are positive-definite (as they will be for sensible theories) and the scalar potential is non-positive, then the argument in the Appendix proves that $A_0 \leq 0$. This rules out $A_0 > 0$. Further, if $A_0 = 0$ then $k^i = 0$, the Maxwell fields must vanish, the scalars must take constant values such that the potential vanishes (if the potential is strictly negative then this is already a contradiction), and the near-horizon geometry must be flat: a direct product of $R^{1,1}$ and T^{D-2}. Hence this case can only arise for toroidal horizon topology, and is therefore excluded if we assume that the horizon is non-toroidal. This concludes the proof of Theorem 1.

Finally, we return to the special case in which the near-horizon metric has AdS_3 symmetry (equation (21)). An obvious question is whether the Maxwell fields in the general theory considered

[4] The near-horizon limit eliminates any r-component of F^I except for F^I_{vr}.

here also inherit the symmetries of AdS_3. We will focus on the $D = 5$ case. The $D = 5$ Einstein equation is:

$$R_{\mu\nu} = \frac{1}{2} f_{AB} \partial_\mu \phi^A \partial_\nu \phi^B + \frac{1}{3} V(\phi) g_{\mu\nu} + \frac{1}{2} g_{IJ} F^I_{\mu\alpha} F_\nu^{J\alpha} - \frac{1}{12} g_{IJ} F^I_{\alpha\beta} F^{J\alpha\beta} g_{\mu\nu}. \tag{28}$$

For a metric of the form (21), $R_{vv} = 0$ automatically, so the (vv) component of the Einstein equation becomes

$$g^2 r^2 \left(F^I_{\rho 1} F^J_{\rho 1} g_{IJ} + F^I_{vr} F^J_{vr} g_{IJ} \Gamma^{-1} \right) = 0 \tag{29}$$

and hence, assuming g_{IJ} is positive definite, $F^I_{vr} = 0$ and $F^I_{\rho 1} = 0$. Therefore the Maxwell field simplifies to

$$F^I = F^I_{\rho 2} d\rho \wedge dx^2 \tag{30}$$

which is manifestly $O(2,2) \times U(1)$ invariant. The scalars are trivially invariant under this symmetry, and thus we learn that in this special case the full solution must be $O(2,2) \times U(1)$ invariant.

In this special case with AdS_3 symmetry, the horizon topology must be $S^1 \times S^2$. The near-horizon geometry is generically a warped product of AdS_3 and S^2 (with the warp factor a function of the polar angle on S^2). However, if Γ is a constant, then one can argue (using the equations of motion) that the near-horizon geometry is a *direct* product of (locally) AdS_3 and S^2 with constant scalars. The near-horizon of the supersymmetric black ring [37] is in this class. In a recent investigation of the existence of asymptotically AdS_5 supersymmetric black rings, we found a near-horizon geometry with AdS_3 symmetry and non-trivial warping but it was not possible to eliminate a conical singularity from the S^2 [39].

2.5 Higher derivative corrections

Much of the recent interest in the attractor mechanism derives from its validity in the presence of higher-derivative terms. It is therefore of interest to examine whether such terms affect our result. In this section we will prove Theorem 2 stated in the Introduction. The following lemma will prove useful:

Lemma 3. Consider the $O(2,1) \times U(1)^{D-3}$-symmetric near-horizon space-time (14). Let J be a conserved current invariant under $SO(2,1) \times U(1)^{D-3}$. Assume that $m_i \equiv \partial/\partial x^i$ vanishes somewhere in the near-horizon geometry for some i (as will be the case if the horizon topology is non-toroidal). Then $J^\rho = 0$.

Proof. $SO(2,1) \times U(1)^{D-3}$ symmetry implies that

$$J = J^\rho(\rho) \frac{\partial}{\partial \rho} + J^i(\rho) \frac{\partial}{\partial x^i}. \tag{31}$$

Plugging this into the conservation equation in the background (14) gives

$$0 = \partial_\mu \left(\Gamma \sqrt{\gamma} J^\mu \right) = \frac{d}{d\rho} \left(\Gamma \sqrt{\gamma} J^\rho \right), \tag{32}$$

where $\gamma = \det \gamma_{ij}$. Hence

$$J^\rho = \frac{j}{\Gamma \sqrt{\gamma}}, \tag{33}$$

where j is a constant. But now consider

$$\star \left(m_1 \wedge m_2 \wedge \ldots \wedge m_{D-3} \right) = \Gamma \sqrt{\gamma} dv \wedge dr \wedge d\rho. \tag{34}$$

Let i_J denote the operation of contracting J with the first index of a p-form. Then

$$i_J \star (m_1 \wedge m_2 \wedge \ldots \wedge m_{D-3}) = j dv \wedge dr = j dT \wedge dY. \tag{35}$$

Now evaluate the LHS where m_i vanishes to conclude that $j = 0$ and the result follows.

Now we can examine higher-derivative corrections. Assume that we are dealing with a theory with an action of the form (5) described in Theorem 2. Varying this action will lead to an Einstein equation of the form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} + \sum_{m \geq 1} \lambda^m H^{(m)}_{\mu\nu}, \tag{36}$$

where $T_{\mu\nu}$ is the energy momentum tensor of the 2-derivative part of the action, and $H^{(m)}$ is conserved. The Maxwell equations will take the form

$$\nabla_\mu \left(g_{IJ}(\phi) F^{J\mu\nu} \right) + S^\nu = \sum_{m \geq 1} \lambda^m K^{(m)\nu}, \tag{37}$$

where S^ν is the contribution to the equation of motion from the term $S_{\rm top}$ in the 2-derivative part of the action and $K^{(m)}$ is conserved.

Consider a general $G_2 \times U(1)^{D-3}$ invariant near-horizon solution of these equations of motion. Assume that it is analytic in λ, and possesses the enhanced $SO(2,1) \times U(1)^{D-3}$ symmetry for $\lambda = 0$ (as follows from our analysis of the 2-derivative theory above). We shall present an inductive argument that the solution must be $SO(2,1) \times U(1)^{D-3}$ invariant to all orders in λ.

Introduce the coordinates (v, r, ρ, x^i) as described above. Assume, inductively, that the solution admits enhanced symmetry up to order λ^n, in other words we have $g_{\mu\nu} = \bar{g}_{\mu\nu} + \lambda^{n+1} h_{\mu\nu}$ where the components of $\bar{g}_{\mu\nu}$ are polynomials of degree n in λ, and $\bar{g}_{\mu\nu}$ exhibits symmetry enhancement. Do the same for the other fields. We then have

$$H^{(m)}[g, \phi^A, F^I] = H^{(m)}[\bar{g}, \bar{\phi^A}, \bar{F}^I] + \mathcal{O}(\lambda^{n+1}). \tag{38}$$

Hence

$$R_{\mu\nu}[g] - \frac{1}{2} g_{\mu\nu} R[g] = T_{\mu\nu}[g, \phi^A, F^I] + \sum_{m \geq 1} \lambda^m H^{(m)}_{\mu\nu}[\bar{g}, \bar{\phi^A}, \bar{F}^I] + \mathcal{O}(\lambda^{n+2}). \tag{39}$$

$H^{(m)}[\bar{g}, \bar{\phi}^A, \bar{F}^I]$ must be invariant under $SO(2,1) \times U(1)^{D-3}$ since it is built from the $SO(2,1) \times U(1)^{D-3}$-invariant objects $\bar{g}$, $\bar{\phi}^A$ and $\bar{F}^I$. Furthermore, we know that $H^{(m)}[\bar{g}, \ldots]$ must be conserved with respect to the metric $\bar{g}$. Hence $J_\mu \equiv H^{(m)}_{\mu\nu}[\bar{g}, \ldots] m_i^\nu$ is a $SO(2,1) \times U(1)^{D-3}$-invariant conserved current with respect to $\bar{g}$ so, from Lemma 3 we must have $J_\rho = 0$. It follows that $H^{(m)}_{\rho i}[\bar{g}, \ldots] = 0$. $SO(2,1) \times U(1)^{D-3}$ symmetry implies[5] $H^{(m)}_{\rho v} = k^i r H^{(m)}_{\rho i}$, hence $H^{(m)}_{\rho v}[\bar{g}, \ldots] = 0$ too. Therefore the higher derivative correction to the ρi and ρv components of the Einstein equation is of order $n+2$ in λ.

The scalar fields exhibit enhanced symmetry trivially since they are functions only of ρ in the near-horizon limit. Turning to the vectors, we have

$$K^{(m)}[g, F^I, \phi^A] = K^{(m)}[\bar{g}, \bar{F}^I, \bar{\phi}^A] + \mathcal{O}(\lambda^{n+1}). \tag{40}$$

Our induction hypothesis implies that the vector $K^{(m)}[\bar{g}, \bar{F}^I, \bar{\phi}^A]$ is invariant under $SO(2,1) \times U(1)^{D-3}$. It is also conserved with respect to $\bar{g}$. Hence Lemma 3 implies that $K^{(m)}_\rho[\bar{g}, \bar{F}^I, \bar{\phi}^A] = 0$. Therefore the higher-derivative correction to the ρ-component of the Maxwell equation is of order $n+2$ in λ.

[5] A symmetric tensor invariant under $SO(2,1) \times U(1)^{D-3}$ must have the form $S_{\mu\nu} dx^\mu dx^\nu = S_0(\rho) \left(A_0 r^2 dv^2 + 2 dv dr \right) + S_1(\rho) d\rho^2 + 2 S_i(\rho) d\rho \left(dx^i + k^i r dv \right) + S_{ij}(\rho) \left(dx^i + k^i r dv \right) \left(dx^j + k^j r dv \right)$.

We can now repeat the argument of section (2.4). The only difference is the $\mathcal{O}(\lambda^{n+2})$ corrections to the Einstein and Maxwell equations. The result is that $R_{\rho i}$ and $R_{\rho v}$ are $\mathcal{O}(\lambda^{n+2})$. We conclude that k^i is constant, and $A = A_0\Gamma$, to order $n+1$ in λ, and therefore g and F^I exhibit symmetry enhancement to order $n+1$ in λ.

Having assumed that the fields exhibit enhanced symmetry to order n we see that they must have enhanced symmetry to order $n+1$. We have already proved the result for $n=0$ and hence the result must be valid for all n by induction.

3 Examples

3.1 Determining near-horizon geometries

In this section we will analyse various examples of near-horizon limits of known five-dimensional extremal black holes. These will illustrate some of our general results. We will focus on near-horizon limits of cohomogeneity-2 black holes as these are the most complicated known examples.

The first step in determining a near-horizon geometry is to introduce coordinates regular at the horizon. Rather than giving details for each solution, we shall explain here how to do this for a general class of metrics which encompasses all of the solutions we are interested in. We assume that the black hole metric takes the cohomogeneity-2 form

$$ds^2 = g_{tt}(R,x)dt^2 + 2g_{ti}(R,x)dtd\Phi^i + g_{RR}(R,x)dR^2 + g_{xx}(R,x)dx^2 + g_{ij}(R,x)d\Phi^i d\Phi^j \tag{41}$$

where R is a "radial" coordinate such that $R=0$ is the event horizon, and x is a "polar angle" on the surfaces of constant t and R. All known rotating black hole solutions (including black rings) take the above form. By shifting Φ^i by appropriate (constant) multiples of t, we can ensure that the coordinates are co-rotating, i.e., $\xi \equiv \partial_t$ is null on the horizon. For known extremal black hole solutions we have

$$g_{ti} = f_i(x)R + \mathcal{O}(R^2), \qquad g_{tt} = f_t(x)R^2 + \mathcal{O}(R^3), \qquad g_{RR} = f_R(x)R^{-2} + \mathcal{O}(R^{-1}) \tag{42}$$

for certain functions $f_\mu(x)$.

The above coordinates are not regular on the horizon. Therefore we define new coordinates (v, r, ϕ^i) by

$$R = r, \qquad dt = dv + a(r)dr, \qquad d\Phi^i = d\phi^i + b^i(r)dr \tag{43}$$

where:

$$a(r) = \frac{a_0}{r^2} + \frac{a_1}{r}, \qquad b^i(r) = \frac{b_0^i}{r}. \tag{44}$$

The constants are chosen to make the metric and its inverse analytic at $r=0$. The near-horizon limit is then defined by $v \to v/\epsilon$, $r \to \epsilon r$ and $\epsilon \to 0$, and let us denote the limiting metric by $\hat{g}_{\mu\nu}$. The following components are easily obtained since they are not affected by the transformation to the new coordinates:

$$\hat{g}_{vi} = f_i(x)r, \qquad \hat{g}_{vv} = f_t(x)r^2, \qquad \hat{g}_{ij} = g_{ij}(0,x), \qquad \hat{g}_{xx} = g_{xx}(0,x). \tag{45}$$

Comparing to our standard form for the near-horizon (13) allows one to identify $x^i = \phi^i$, $d\rho^2 = \hat{g}_{xx}(x)dx^2$, and

$$k_i = f_i(x), \qquad A + k^i k_i = f_t(x), \qquad \gamma_{ij} = \hat{g}_{ij}. \tag{46}$$

The vr component of the metric is $g_{vr} = a(r)g_{tt} + b^i(r)g_{ti}$. Taking the near-horizon limit gives:

$$\Gamma = \hat{g}_{vr} = a_0 f_t(x) + b_0^i f_i(x). \tag{47}$$

Hence the near-horizon solution is fully-determined once we know the constants a_0, b_0^i. These can be obtained from regularity of g_{ri} and g_{rr}. Absence of a $1/r$ term in g_{ri} implies that

$$b_0^i = -a_0 \gamma^{ij} f_j(x) = -a_0 k^i. \tag{48}$$

This implies that k^i must be constant. However, this equation is only consistent if $\gamma^{ij} f_j$ is constant. Fortunately this turns out to be true for known solutions. Of course, this is not an accident: we are discussing solutions of the Einstein equation, and we have seen in previous sections that the Einstein equation implies that k^i must be constant.

Absence of a $1/r^2$ term in g_{rr} implies that[6]

$$a_0^2 = \frac{f_R(x)}{k^i k_i - f_t(x)}. \tag{49}$$

This determines a_0 up to a sign, and hence b_0^i and Γ are also determined up to the same sign. We choose the sign such that $\Gamma > 0$, which ensures that we are dealing with a future, rather than past, horizon. The final piece of near-horizon data, A is then determined from (46). Combining this with (47) and (48) gives:

$$A = a_0^{-1} \Gamma. \tag{50}$$

Equations (48) and (50) imply that the near-horizon metric is of the form (14) with $A_0 = a_0^{-1}$. Thus we see the enhancement of symmetry, which we derived earlier using more general arguments.

3.2 S^3 topology black holes

The simplest cohomogeneity-2 black hole with an S^3 topology horizon is the doubly spinning Myers-Perry solution [21] (with $a \neq b$, where a, b are the rotation parameters). Using the general formalism developed in the previous section we can calculate the near-horizon limit of the extremal doubly spinning Myers-Perry black hole with $a \neq b$. Without loss of generality we choose $0 < a < b$. After a tedious calculation, we find that the near horizon limit can be written in the form (14) where:

$$\begin{aligned}
\gamma_{ab} dx^a dx^b &= \sigma(\theta)^2 d\theta^2 \\
&+ \frac{(a+b)^2}{\sigma(\theta)^2} \left(b\cos^2\theta (b + a\cos^2\theta) d\psi^2 + 2r_0^2 \cos^2\theta \sin^2\theta d\phi d\psi + a \sin^2\theta (a + b\sin^2\theta) d\phi^2 \right) \\
\Gamma &= \frac{\sigma(\theta)^2}{(a+b)^2}, \qquad A_0 = -\frac{4}{(a+b)^2}, \qquad k^i \frac{\partial}{\partial x^i} = \frac{2r_0}{b(a+b)^2} \frac{\partial}{\partial \psi} + \frac{2r_0}{a(a+b)^2} \frac{\partial}{\partial \phi}
\end{aligned} \tag{51}$$

where the coordinates on the horizon are $x^a = (\theta, \phi, \psi)$ and $\sigma(\theta)^2 = r_0^2 + a^2 \cos^2\theta + b^2 \sin^2\theta$, $r_0^2 = ab$.

3.3 $S^1 \times S^2$ topology black holes

Vacuum solutions. There are two known vacuum solutions with horizon topology $S^1 \times S^2$. These are the (boosted) Kerr string and the black ring [22, 23][7]. The 3-parameter black ring solution admits an extremal limit [23]. In this section, we will use the formalism described above to show that the near-horizon geometry of an extremal black ring is globally isometric to that of an extremal Kerr string.

Let us begin with the boosted Kerr string. To construct this solution one takes the direct product Kerr $\times S^1$, and write the metric on S^1 as dz^2. Now perform a boost $(t, z) \to (\cosh\beta t + \sinh\beta z, \sinh\beta t + \cosh\beta z)$, where t is the time coordinate in which the Kerr metric is at rest at

[6] There will also be a $1/r$ divergence in g_{rr}. Eliminating this determines a_1. However, we do not need to know a_1 to determine the near-horizon geometry since a dr^2/r term vanishes in the near-horizon limit.

[7] Only the black ring is asymptotically flat.

infinity. After taking the extremal limit, we find that the near-horizon geometry has the expected $O(2,1)\times U(1)^2$ symmetry. The near-horizon data is:

$$\begin{aligned}\gamma_{ab}dx^a dx^b &= a^2(1+\cos^2\theta)d\theta^2+\frac{4a^2\sin^2\theta}{1+\cos^2\theta}\left(d\Phi+\frac{\sinh\beta}{2a}dz\right)^2+\cosh^2\beta dz^2, &(52)\\ \Gamma &= \frac{1+\cos^2\theta}{2\cosh\beta},\qquad A_0=-\frac{1}{2a^2\cosh\beta},\qquad k^i\partial_i=\frac{1}{2a^2\cosh\beta}\partial_\Phi &(53)\end{aligned}$$

where (θ, z, Φ) are the coordinates on the horizon[8] and a is the Kerr angular momentum parameter. Note that Φ has period 2π, whereas z may have any period Δz. This near-horizon solution thus depends on three parameters $(a, \beta, \Delta z)$.

Let us now turn to the extremal limit of the recently discovered vacuum black ring with two independent angular momenta [23]. The solution has three parameters (k, λ, ν) and the extremal limit is reached when $\nu = \lambda^2/4$, and $0 \leq \lambda \leq 2$. After an involved calculation, we find the near horizon can be written in the form (14) with[9]

$$\begin{aligned}\gamma_{ab}dx^a dx^b &= \frac{8\lambda^2k^2H(x)}{(\lambda x+2)^4(1-x^2)(4-\lambda^2)}dx^2\\ &+ \frac{32\lambda^2k^2(1-x^2)}{(4-\lambda^2)H(x)}\left(d\phi+\frac{\lambda^2+8\lambda+4}{4\lambda}d\psi\right)^2+\frac{4(2+\lambda)^2k^2}{(2-\lambda)^2}d\psi^2 &(54)\\ \Gamma &= \frac{k\lambda^2H(x)}{2(2+\lambda x)^2(2+\lambda)^2},\qquad A_0=-\frac{(2-\lambda)^2}{16k},\qquad k^i\partial_i=\frac{(2-\lambda)^2}{16k}\partial_\phi,\end{aligned}$$

where $H(x) = (\lambda^2 + 4)(1 + x^2) + 8\lambda x$, and (x, ψ, ϕ) are coordinates on the horizon such that $-1 \leq x \leq 1$ and ψ, ϕ both have period 2π.

We can now prove that this 2-parameter near-horizon geometry is globally isometric to a special case of the 3-parameter Kerr string near-horizon geometry. First, it is necessary to rescale the v coordinate of the boosted Kerr string: $v \to Cv$. The following coordinate transformation then proves they are globally equivalent:

$$\cos\theta=\frac{2x+\lambda}{2+\lambda x},\qquad \Phi=\phi+\psi,\qquad z=\frac{\sqrt{2}k(2+\lambda)}{(2-\lambda)}\psi \tag{55}$$

provided that the Kerr string parameters are chosen to be:

$$a^2=\frac{8k^2\lambda^2}{(4-\lambda^2)^2},\qquad \sinh^2\beta=1,\qquad \Delta z=\frac{2\sqrt{2}\pi k(2+\lambda)}{2-\lambda}\qquad C=\frac{\sqrt{2}k\lambda^2}{(2+\lambda)^2}. \tag{56}$$

Dipole rings. The dipole ring solution of $D = 5$ Einstein-Maxwell theory [24] admits an extremal limit even though it rotates in only one plane. We will work in the conventions of [46]. After taking the near horizon limit as described above we find the that the resulting solution has $SO(2,1)\times U(1)^2$ symmetry with

$$\begin{aligned}\gamma_{ab}dx^a dx^b &= R^2F(x)H(x)\mu^2\left[\frac{dx^2}{1-x^2}+\frac{1-x^2}{F(x)H(x)^3}d\phi^2\right]+\frac{R^2\lambda(1+\lambda)H(x)}{\mu(1-\lambda)F(x)}d\psi^2\\ \Gamma &= \left(\frac{\mu(1-\lambda)}{\lambda(1+\lambda)}\right)^{1/2}RF(x)H(x),\qquad A_0=-\frac{(1-\lambda)^{1/2}}{R\left(\lambda(1+\lambda)\mu^3\right)^{1/2}}, &(57)\\ k^i\partial_i &= -\frac{(1-\lambda)^{3/2}}{R\lambda(1+\lambda)^{1/2}}\partial_\psi,\end{aligned}$$

[8] This z is not the same as the initial one: it has suffered from two shifts, a constant one in the t direction (to go to co-rotating coordinates) and a singular one in the r direction (to go to coordinates regular at the horizon).

[9] Note that unlike in [23] we work with a mostly positive signature, and we call ψ the angle along the S^1 of the ring.

where $F(x) = 1 + \lambda x$ and $H(x) = 1 - \mu x$, with $0 \leq \lambda, \mu < 1$. The gauge potential in this case is simply given by:

$$A = \left(\frac{1-\mu}{1+\mu}\right)^{1/2} \frac{\mu R(1+x)}{H(x)} d\phi \tag{58}$$

3.4 Analytic continuation

We have seen that near-horizon solutions necessarily possess an $SO(2,1)$ symmetry. For non-static black holes, $SO(2,1)$ has 3-dimensional orbits, which have the form of a line, or circle, bundle over AdS_2. In this section, we note that near-horizon solutions can sometimes be analytically continued so that AdS_2 becomes S^2 and $SO(2,1)$ becomes $SU(2)$ acting on a circle bundle over S^2 (this bundle is just S^3 in the case we shall discuss). This generalizes the analytic continuation relating solutions with AdS_3 and S^3 symmetries that has been studied in [47, 48].

Consider first the near-horizon geometry of the extremal dipole ring discussed above. One can analytically continue this near-horizon geometry to obtain an $SU(2)$-symmetric non-extremal Kaluza-Klein black hole solution. To see this, transform from (v, r, ψ) to the global coordinates (T, Y, χ) defined earlier (17, 18) and continue $Y \to ig^{-1}\cos\theta$, $\chi \to ik^{\psi}g^{-2}\chi$ and rescale $\phi \to t/R$, $T \to g^{-1}T$, where (g, k^{ψ}) are given in (57). The result is:

$$\begin{aligned} ds^2 &= -\frac{\mu^2(x^2-1)}{(\mu x-1)^2}dt^2 + \frac{R^2F(x)(\mu x-1)dx^2}{x^2-1} + R^2\mu^2(\mu x-1)F(x)(d\theta^2 + \sin^2\theta dT^2) \\ &+ R^2\mu^2(1-\lambda^2)\frac{(\mu x-1)}{F(x)}(d\chi + \cos\theta dT)^2, \end{aligned} \tag{59}$$

It is easy to see one may write the (θ, T, χ) part of the metric in terms of left invariant one-forms on $SU(2)$, so that $\sigma_3 = d\chi + \cos\theta dT$ and $\sigma_1^2 + \sigma_2^2 = d\theta^2 + \sin^2\theta dT^2$. Now make the following coordinate transformation: $r = R\sqrt{\lambda\mu}(\mu x - 1)$. This gives:

$$ds^2 = -Vdt^2 + \frac{U}{V}dr^2 + Ur^2(\sigma_1^2 + \sigma_2^2) + W\sigma_3^2, \qquad F = d\left[\frac{\sqrt{r_+r_-}}{r}dt\right] \tag{60}$$

where

$$V = \frac{(r-r_+)(r-r_-)}{r^2}, \qquad U = \frac{r+r_0}{r}, \qquad W = \frac{(r_+ + r_0)(r_- + r_0)}{U} \tag{61}$$

and

$$r_\pm = -R\sqrt{\lambda\mu}(1 \pm \mu), \qquad r_0 = R\sqrt{\lambda\mu}\left(1 + \frac{\mu}{\lambda}\right). \tag{62}$$

This is the Kaluza-Klein black hole solution discussed in [49].[10]

Next we shall show that the near-horizon geometry (51) of the extremal Myers-Perry black hole is related by analytic continuation to the non-extremal self-dual Myers-Perry solution, which has metric

$$ds^2 = -f^2dt^2 + g^2dr^2 + \frac{h^2}{4}(\sigma_3 - \Omega dt)^2 + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2), \tag{63}$$

$$g^{-2} = 1 - \frac{2M}{r^2} + \frac{2M\alpha^2}{r^4}, \qquad h^2 = r^2\left(1 + \frac{2M\alpha^2}{r^4}\right), \qquad f = \frac{r}{hg}, \qquad \Omega = \frac{4M\alpha}{r^2h^2} \tag{64}$$

where σ_i are again left-invariant one-forms on $SU(2)$. To do this, we will work backwards from the self-dual Myers-Perry solution. First we make the following coordinate change: $r^2 = 4\Gamma/C^2$, $y =$

[10] In order for this specetime to be regular on and outside the horizon $r = r_+$ we require $r_+ > 0$, $r_+ > r_-$ and $r_+ > -r_0$. These conditions can be fulfilled in two ways: (i) $R < 0$, $\mu > 0$, $\lambda > 1$ or (ii) $R > 0$, $\mu < -1$, $\lambda < 0$.

$\cos\theta$ where we denote the Euler angles by (θ, τ, ψ) so $\sigma_3 = d\psi + \cos\theta d\tau$ and $\sigma_1^2 + \sigma_2^2 = d\theta^2 + \sin^2\theta d\tau^2$. Now perform the following analytic continuation: $y \to iy$, $C^2 \to -C^2$ and $\alpha \to i\alpha$. The resulting metric is:

$$ds^2 = \frac{\Gamma}{C^2}\left[-(1+y^2)d\tau^2 + \frac{dy^2}{1+y^2} + \frac{A(\Gamma)}{\Gamma^2}(d\psi + yd\tau + \omega(\Gamma)dt)^2\right] + \frac{\Gamma d\Gamma^2}{4P(\Gamma)} + \frac{4P(\Gamma)dt^2}{C^2 A(\Gamma)} \tag{65}$$

where

$$P(\Gamma) = -\frac{C^2}{4}\Gamma^2 - \frac{MC^4}{8}\Gamma + \frac{M\alpha^2 C^6}{32}, \tag{66}$$

$$A(\Gamma) = \Gamma^2 - \frac{M\alpha^2 C^4}{8}, \qquad \omega(\Gamma) = -\frac{M\alpha C^4}{4A(\Gamma)}. \tag{67}$$

Using the inverse of the AdS_2 coordinate transformation used earlier (17, 18), with $C = g$, $y = gY$, $\tau = gT$, $\psi = g^2\chi$, and letting $x^2 = t$, we get the following geometry:

$$ds^2 = \Gamma\left[-C^2 r^2 dv^2 + 2dvdr + C^2\frac{A(\Gamma)}{\Gamma^2}(dx^1 + rdv + C^{-2}\omega(\Gamma)dx^2)^2\right] + \frac{\Gamma d\Gamma^2}{4P(\Gamma)} + \frac{4P(\Gamma)(dx^2)^2}{C^2 A(\Gamma)}. \tag{68}$$

It is now apparent that this metric looks like a near-horizon geometry. The corresponding metric on the horizon at $r = 0$ is:

$$\gamma_{ab}dx^a dx^b = \frac{C^2 A(\Gamma)}{\Gamma}(dx^1 + C^{-2}\omega(\Gamma)dx^2)^2 + \frac{\Gamma d\Gamma^2}{4P(\Gamma)} + \frac{4P(\Gamma)(dx^2)^2}{C^2 A(\Gamma)}. \tag{69}$$

In order to prove that this near-horizon geometry is the near-horizon limit (51), it is necessary that the horizon metric (69) can be made globally regular with S^3 topology. Enforcing compactness and regularity allows us to deduce the required coordinate change to prove the equivalence of the two metrics. The calculation is similar to that done in [39] and we omit the details. The explicit transformations $(\Gamma, x^i) \to (\theta, \psi, \phi)$ are given by:

$$\Gamma = \frac{\sigma(\theta)^2}{(a+b)^2}, \qquad x^1 = \frac{r_0(a+b)^2}{2(a-b)}(\psi - \phi), \qquad x^2 = \frac{(a+b)}{(a-b)}(a\phi - b\psi), \tag{70}$$

and the parameters are related by

$$C^2 = \frac{4}{(a+b)^2} \qquad M = -\frac{(a+b)^2}{2} \qquad \alpha^2 = ab. \tag{71}$$

With these identifications it is straightforward to confirm that the two near-horizon metrics (51) and (68) are identical (although the mass M has to be taken negative). Notice that the (Γ, x^i) coordinate system is simpler than the "natural" coordinates one obtains from taking the near-horizon limit. This is actually the same coordinate system encountered in [39] for the near-horizon geometry of a supersymmetric AdS_5 doubly spinning black hole [43].

The above analytic continuation can be generalised to other near horizon geometries with spherical topology horizons, such as Myers-Perry-AdS [42] and charged versions of this [43, 44]. In particular, we find that the near horizon limit of the Chong *at al* supersymmetric AdS_5 black hole [39] analytically continues to the Klemm-Sabra "time-machine" [45]. This implies that the number of supersymmetries of the Chong *et al* solution is enhanced from two to four in the near horizon limit.

We should point out that analytic continuation does not always lead to a stationary black hole solution. For example, if one starts from the near-horizon Kerr solution then one ends up with a special case (vanishing mass) of the Lorentzian Taub-NUT solution. This has $SO(3) \times U(1)$ symmetry, where $SO(3)$ has 3-dimensional orbits and acts non-trivially on the time coordinate,

giving rise to a non-vanishing NUT charge. There is no way of avoiding this in $D = 4$ (except for giving up stationarity). In $D = 5$ the near-horizon symmetry group is $SO(2,1) \times U(1)^2$. This group has 4-dimensional orbits, but the orbits of $SO(2,1)$ are only 3-dimensional. This gives the possibility of analytically continuing in such a way that the new "time" direction lives within the surfaces of homogeneity, but is not acted on by $SU(2)$ (or $SO(3)$), thereby avoiding NUT charge. This is precisely what we have done above.

3.5 The general solution for a vacuum near-horizon geometry

Lemma 1 tells us that a Ricci flat near-horizon geometry must be of the form (14). In this section we shall show that one can determine, at least implicitly, the general Ricci-flat solution of this form. We will assume $A_0 \neq 0$ since otherwise $k = 0$ and the geometry is static, which leads to a trivial near-horizon geometry [40] (assuming a compact horizon). Our method is motivated by the analytic continuation described above: continuation of a $SO(2,1) \times U(1)^{D-3}$-invariant vacuum near-horizon solution gives a vacuum solution with symmetry group $\sim SO(3) \times U(1)^{D-3}$. This is the symmetry of a stationary, spherically symmetric solution of D-dimensional Kaluza-Klein theory. All such solutions were obtained in [35] using a method introduced in [41]. Hence, by repeating the analysis of [41, 35] we can determine the general vacuum near-horizon solution.

We start with a reduction to three dimensions on the $D-3$ commuting Killing vectors $\xi_i \equiv \partial/\partial x^i$. For convenience, introduce a new coordinate σ by $d\sigma^2 = \gamma d\rho^2$ where $\gamma = \det \gamma_{ij}$ and let $f^2 = \gamma\Gamma$. The full spacetime metric $g_{\mu\nu}$ may be expressed in terms of 3d data: a set of functions γ_{ij} , a set of one-forms (twist vectors) $\Omega_i = \star(d\xi_i \wedge \xi_1 \wedge \xi_2 \wedge \cdots \xi_{D-3})$, and an induced metric $h_{ab} = \gamma(g_{ab} - \gamma^{ij}\xi_{ia}\xi_{jb})$ where the indices a, b run over all coordinates except x^i [35]. T The vacuum equations imply that the twist vectors are closed, $\Omega_i = d\omega_i$ (ω_i are called the twist potentials), and leads to a 3d sigma model with equations of motion:

$$R_{ab}(h) = \frac{1}{4}\mathrm{Tr}(\partial_a \chi \chi^{-1} \partial_b \chi \chi^{-1}), \qquad D^a(\partial_a \chi \chi^{-1}) = 0. \tag{72}$$

where D is the covariant derivative wrt h and χ is a symmetric $(D-2) \times (D-2)$ unimodular matrix [41, 35]:

$$\chi = \begin{pmatrix} \gamma^{-1} & -\gamma^{-1}\omega_i \\ -\gamma^{-1}\omega_i & \gamma_{ij} + \gamma^{-1}\omega_i\omega_j \end{pmatrix}. \tag{73}$$

In the case we are considering:

$$h_{ab}dx^a dx^b = d\sigma^2 + f^2(\sigma)(A_0 r^2 dv^2 + 2dvdr), \qquad \omega_i = \int d\sigma k_i(\sigma)\gamma f^{-2}(\sigma) \tag{74}$$

and the non-vanishing components of the Ricci tensor of h are:

$$R_{vr} = -\frac{1}{2}\left(\frac{d^2 f^2}{d\sigma^2} - 2A_0\right), \qquad R_{\sigma\sigma} = -\frac{1}{2f^4}\left[2f^2\frac{d^2 f^2}{d\sigma^2} - \left(\frac{df^2}{d\sigma}\right)^2\right]. \tag{75}$$

Since χ is independent of v, from the sigma model equations we see that $R_{vr}(h) = 0$ and thus we can integrate to get f^2:

$$f^2 = A_0[(\sigma - b)^2 - a^2/4] \tag{76}$$

where a, b are constants ($f^2 \geq 0$ implies $a^2 \geq 0$ as $A_0 < 0$). The field equation for χ simplifies as χ only depends on σ:

$$\frac{d}{d\sigma}\left(\frac{d\chi}{d\sigma}\chi^{-1}f^2\right) = 0 \tag{77}$$

which can be integrated:

$$\chi = \exp[\mu R(\sigma)]\chi_0, \qquad R(\sigma) = \int \frac{d\sigma}{f^2} \tag{78}$$

where μ is an arbitrary constant traceless matrix, χ_0 is a constant unimodular symmetric matrix and $\chi_0 \mu^T = \mu \chi_0$. The $\sigma\sigma$ component of the Ricci equation then implies $\mathrm{Tr}\mu^2 = 2A_0^2 a^2$. Performing the integration explicitly gives:

$$R(\sigma) \;=\; \frac{1}{A_0 a} \log \left| \frac{\sigma - b - a/2}{\sigma - b + a/2} \right| . \tag{79}$$

Thus we have completely determined the 3d data h_{ab} and χ, and hence a general vacuum near-horizon geometry, in terms of the constant matrices μ, χ_0 and two other integration constants a, b subject to the constraints derived above. Note that $\chi \to M\chi M^T$ where M is in $SL(D-2, R)$ leaves the 3d field equations invariant. On our solution for χ this freedom reads $\chi_0 \to M\chi_0 M^T$ and $\mu \to M\mu M^{-1}$. However, solutions to the 3d equations related by this symmetry will not in general lead to equivalent spacetime geometries in D dimensions. However, a subgroup of these transformations which does lead to equivalent D dimensional geometries is the $GL(D-3, R)$ group which mixes the the $D-3$ Killing vectors.

The above analysis is local: compactness of the horizon has not been enforced. This will impose further restrictions. In the $D = 4$ case, it is known that the general axisymmetric vacuum near-horizon solution with S^2 topology is that of the extremal Kerr solution [50].

4 Discussion

There are various ways in which our results could be extended. For example, Theorem 2 assumes the black hole is a black hole to lowest order (i.e. within Einstein gravity). Thus our result does not apply to "small" black holes which are black holes only when higher derivative terms are taken into account. It would be nice to extend our proof to remove this assumption. However, this may depend on the details of precisely which higher-derivative terms are required.

In five dimensions, our results assume two rotational symmetries, whereas the "stationary implies axisymmetric" theorem guarantees only one. It would be interesting to see whether one could extend our results in five dimensions by removing the assumption of this extra rotational symmetry.

We have considered a theory of gravity coupled to abelian vectors and uncharged scalars. Can our results be generalized to theories with non-abelian vectors and/or charged scalars?

We have commented on the special case of a static near-horizon geometry with local AdS_3 symmetry. As we have seen, in this case the near-horizon geometry is a warped product of AdS_3 and S^2 and the horizon topology $S^1 \times S^2$, i.e., a black ring. Of particular relevance to string theory is the question of whether this structure will be preserved by higher-derivative corrections, or whether it will be broken to the global AdS_2 symmetry that we have shown must always exist.[11] One can attempt to modify the argument of section 2.5 to prove that AdS_3 symmetry must exist to all orders in λ if it exists for $\lambda = 0$ but this does not work.[12] The problem is that AdS_2-symmetry is a consequence of the equations of motion, whereas AdS_3 appears to be an "accident" that arises when a non-static black ring solution happens to have a static near-horizon geometry. In general, there is no reason why this accidental feature should persist in the presence of higher-derivative terms.

[11] See [11] for a complementary discussion of this point.

[12] To see what goes wrong, consider (for simplicity) a theory of pure gravity. The argument of section 2.5 consisted of 2 steps. First, we showed that if the metric is $SO(2,1)$-invariant to order n in λ then the RHS of the Einstein equation is $SO(2,1)$-invariant to order $n+1$, hence the Einstein tensor is $SO(2,1)$-invariant to order $n+1$. Second, we showed that this implies that the metric is $SO(2,1)$-invariant to order $n+1$. The second step doesn't work for $O(2,2)$ symmetry, i.e., an $O(2,2)$-symmetric Einstein tensor does not imply an $O(2,2)$-symmetric metric. This is obvious even at zeroth order: the RHS of the Einstein equation is zero, which is obviously $O(2,2)$-symmetric, but this does not imply that any vacuum near-horizon metric must be $O(2,2)$-symmetric, in fact none is! (This follows from [40].)

This conclusion may be modified if one imposes additional symmetries on the solution. For example, in supergravity theories one can impose the condition that the black hole be supersymmetric, and supersymmetry may then explain the "accidental" AdS_3. For example, in minimal 5d supergravity, the only supersymmetric near-horizon solution with $S^1 \times S^2$ topology is $AdS_3 \times S^2$ [19], corresponding to the near-horizon geometry of a supersymmetric black ring [37]. It would be interesting to extend the classification [19] of supersymmetric near-horizon geometries to include higher-derivative terms to see whether this conclusion persists in a more general theory.

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A Excluding near-horizon geometries with $A_0 \geq 0$

In this section we wish to show that there are no near-horizon geometries with compact horizons with $A_0 \geq 0$ in the general second order theory (2). We will assume $V(\phi) \leq 0$, and that f_{AB}, g_{IJ} are positive definite.

For reference, the metric is:

$$ds^2 = \Gamma(\rho)[A_0 r^2 dv^2 + 2dvdr] + d\rho^2 + \gamma_{ij}(\rho)(dx^i + k^i r dv)(dx^j + k^j r dv) \tag{80}$$

and recall that $\Gamma > 0$. The Maxwell fields are given by

$$F^I = \Delta^I(\rho)\, dv \wedge dr + B^I_i(\rho)\, d\rho \wedge (dx^i + k^i r dv) \tag{81}$$

where we have defined the functions $\Delta^I \equiv F^I_{vr}(\rho)$ and $B^I_i \equiv F^I_{\rho i}(\rho)$ for notational convenience.

We can now generalise an argument of Gibbons [51]. For the above metric, we have

$$R_{vr} = A_0 + \frac{k^i k_i}{2\Gamma} - \frac{1}{2}\left[\Gamma'' + \frac{\Gamma'\gamma'}{2\gamma}\right] = A_0 + \frac{k^i k_i}{2\Gamma} - \frac{1}{2}\hat{\nabla}^2\Gamma, \tag{82}$$

where $\hat{\nabla}$ is the metric connection on H. The vr component of the Einstein equation is

$$A_0 - \frac{1}{2}\hat{\nabla}^2\Gamma = -\frac{k^i k_i}{2\Gamma} + \frac{1}{D-2}\Gamma V - \frac{(D-3)}{2(D-2)}\Gamma^{-1} g_{IJ}\Delta^I\Delta^J - \frac{1}{2(D-2)}\Gamma g_{IJ}\gamma^{ij} B^I_i B^J_j. \tag{83}$$

Note that our assumptions imply that the RHS is non-positive: this is a consequence of this theory obeying the strong energy condition. Integrate this equation over H to conclude that $A_0 \leq 0$ which rules out the case $A_0 > 0$. Further, if $A_0 = 0$ we must have $k^i = 0$ and $\Delta^I \equiv 0$, $B^I_i \equiv 0$, i.e., the Maxwell fields vanish, and $V = 0$ everywhere. If the theory has $V < 0$ then this is a contradiction and we are done. Otherwise we conclude that the scalars must take values such that the potential is at its maximum $V = 0$ everywhere. The above equation then tell us that Γ is harmonic, and hence must be constant. Therefore the geometry is a direct product of $R^{1,1}$ and H.

The next step is to show that the scalars must be constant.[13] The scalar equation of motion admits the integral

$$\frac{1}{2} f_{AB}\phi^{A'}\phi^{B'} = E, \tag{84}$$

[13] If the assumption that the scalars are invariant under the rotational symmetries were relaxed then this needn't be true [52].

where E is a (non-negative) constant and we have used $V = 0$. Hence if we can show $E = 0$ then the scalars are constant. From the Einstein equation we find

$$E = R_{\rho\rho} - \gamma^{ij} R_{ij} = \frac{1}{4} \left[\frac{\gamma'^2}{\gamma^2} - \gamma^{ij} \gamma^{kl} \gamma'_{ik} \gamma'_{jl} \right], \tag{85}$$

where γ^{ij} is the inverse of γ_{ij}, $\gamma \equiv \det \gamma_{ij}$, and we have used the fact that Γ is constant. If $D = 4$ then the RHS vanishes identically and hence $E = 0$. For $D = 5$ we can argue as follows. As explained in the main text, ρ takes values on an interval such that γ is positive on the interior of the interval and vanishes at the endpoints. Hence there must be a point in the interior of this interval for which $\gamma' = 0$. Evaluate the above equation at this point. The RHS is manifestly non-positive, but the LHS is non-negative. Hence we must have $E = 0$. Therefore the scalars are constant.

We have shown that the scalars are constant, the Maxwell fields vanish, $V = 0$, and Γ is constant. Hence the Einstein equation reduces to the vacuum Einstein equation. The metric is a direct product of 2-dimensional flat space with H so the Einstein equation implies that H is Ricci-flat and hence flat (as H is 2 or 3-dimensional). Therefore H must be a torus, contradicting the assumption of Theorem 1.

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